Lesson 16. Multivariable Functions

1 This lesson...

- How do functions of many variables work? What do they look like graphically?

2 Functions of 2 variables

- A **function f of 2 variables**
  - takes an ordered pair (x, y) of real numbers as input
  - outputs a unique real number $f(x, y)$
- The **domain** D of f is the set of allowable inputs to f
  - If f is given by a formula and its domain is not explicitly specified, then the domain of f is the set of all (x, y) for which the formula is well-defined
- The **range** of f is the set of values that f takes on

Example 1. Let $f(x, y) = \dfrac{\sqrt{x + y + 1}}{x - 1}$.

a. What is $f(3, 2)$?
b. What is the domain of f?

Example 2. In 1928, using economic data published by the government, Charles Cobb and Paul Douglas modeled production output $P(L, K)$ as a function of the amount of labor involved L and the amount of capital invested K:

$$P(L, K) = 1.01L^{0.75}K^{0.25}$$

This function (in a more general form) is known as the *Cobb-Douglas production function*.

Find $P(120, 20)$. In words, what does $P(120, 20)$ mean?

- Functions are not always represented by explicit formulas, as the next example shows

Example 3. The *wind-chill index* $W(T, v)$ is a subjective temperature that is a function of the actual temperature T (in °C) and wind speed v (in km/h), as given by the table below:

Wind speed (km/h)

Actual temperature (°C) T \ v	5	10	15	20	25	30	40	50	60	70	80
5	4	3	2	1	1	0	−1	−1	−2	−2	−3
0	−2	−3	−4	−5	−6	−6	−7	−8	−9	−9	−10
−5	−7	−9	−11	−12	−12	−13	−14	−15	−16	−16	−17
−10	−13	−15	−17	−18	−19	−20	−21	−22	−23	−23	−24
−15	−19	−21	−23	−24	−25	−26	−27	−29	−30	−30	−31
−20	−24	−27	−29	−30	−32	−33	−34	−35	−36	−37	−38
−25	−30	−33	−35	−37	−38	−39	−41	−42	−43	−44	−45
−30	−36	−39	−41	−43	−44	−46	−48	−49	−50	−51	−52
−35	−41	−45	−48	−49	−51	−52	−54	−56	−57	−58	−60
−40	−47	−51	−54	−56	−57	−59	−61	−63	−64	−65	−67

a. Find $W(-15, 40)$. In words, what does $W(-15, 40)$ mean?
b. Define the function $h(T) = W(T, 40)$. Describe the behavior of h.

3 Functions of n variables

- A **function f of n variables**
  - takes an ordered tuple $(x_1, \ldots, x_n)$ of real numbers as input
  - outputs a unique real number $f(x_1, \ldots, x_n)$

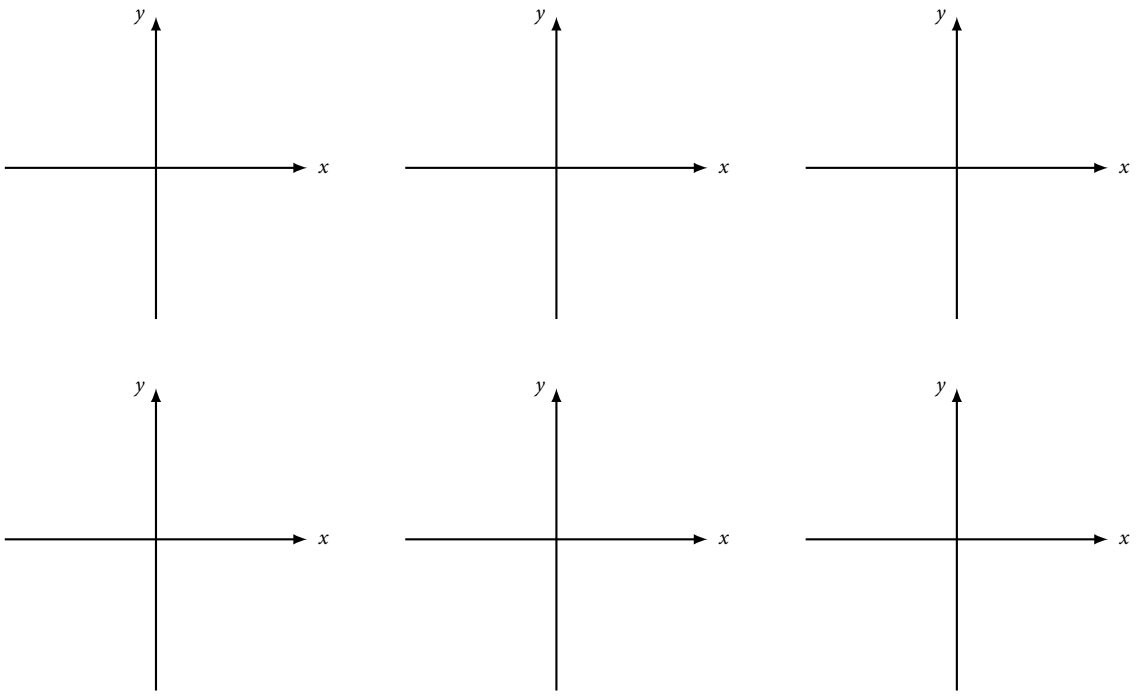
Example 4. Anteater-Bugs produces n types of beers. It costs c_i to produce one bottle of type i beer ($i = 1, \ldots, n$). Let $C(x_1, x_2, \ldots, x_n)$ be the total cost of producing x_1 bottles of type 1 beer, x_2 bottles of type 2 beer, …, and x_n bottles of type n beer.

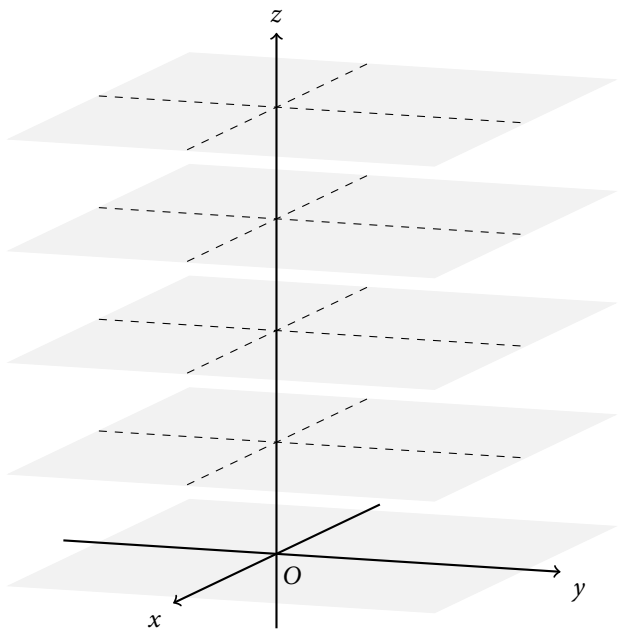
Write a formula for $C(x_1, x_2, \ldots, x_n)$.

4 Graphs of functions in 2 variables

- Let f be a function of 2 variables with domain D
- The **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D

Example 5. Sketch the graph of $f(x, y) = x^2 + 2y^2 + 1$ by first drawing its traces for $z = -1, 0, 1, 2, 3, 4$.

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